# Math 246C Lecture 21 Notes

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May 21, 2019

## 1 The $\overline{\partial}$-Equation, the Hartogs Extension Theorem, and Regularization of Subharmonic Functions

### 1.1 Compactly supported solutions of the $\overline{\partial}$-equation

**Theorem 1.1.** *Let $f_j \in C_0^k(\mathbb{C}^n)$ for $1 \leq j \leq n$ and $n > 1$ be such that $\overline{\partial} f = 0$. Then the equation $\overline{\partial} u = f$ has a unique solution $u \in C_0^k(\mathbb{C}^n)$.*

*Proof.* Consider $\frac{\partial u}{\partial \overline{z}_j}$ for $1 \leq j \leq n$. Define

$$u(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f_1(\zeta_1, z_2, \dots, z_n)}{\zeta_1 - z_1} \, L(d\zeta_1).$$

Then $u \in C^k(\mathbb{C}^n)$, and $\frac{\partial u}{\partial \overline{z}_1} = f_1$. When $j > 1$, we have by the compatibility conditions that

$$\frac{\partial u}{\partial \overline{z}_j} = -\frac{1}{\pi} \iint \frac{\frac{\partial f_1}{\partial \overline{z}_j}(\zeta_1, z_2, \dots, z_n)}{\zeta - z_1} L(d\zeta_1) = \frac{1}{\pi} \iint \frac{\frac{\partial f_1}{\partial \overline{z}_1}(\zeta_1, z_2, \dots, z_n)}{\zeta - z_1} L(d\zeta_1) = f_j(z),$$

using Cauchy's integral formula.

We claim that if $n > 1$, then $u$ is compactly supported: If $|z_1| + \cdots + |z_n|$ is large enough, then $u(z) = 0$. On the other hand, $\overline{\partial} u = 0$ on $\mathbb{C}^n \setminus K$, where $K = \bigcup_{i=1}^n \operatorname{supp}(f_i)$ is compact. $u \in \operatorname{Hol}(\mathbb{C}^n \setminus K)$, and if $\Omega$ is the unbounded component, then, as $u(z) = 0$ on some open set in $\Omega$, $u = 0$ in $\Omega$ by the uniqueness of analytic continuation. So $\operatorname{supp}(u) \subseteq K \cup \bigcup \mathcal{M}$, where $M$ is a bounded component of $\mathbb{C}^n \setminus K$. This is bounded, so $u \in C_0^k(\mathbb{C}^n)$. ☐

### 1.2 The Hartogs extension theorem

**Theorem 1.2** (Hartogs extension theorem)**.** *Let $|Omega \subseteq \mathbb{C}^n$ be open with $n > 1$, and let $K \subseteq \Omega$ be compact with $\Omega \setminus K$. Let $u \in \operatorname{Hol}(\Omega \setminus K)$. Then there exists a $U \in \operatorname{Hol}(\Omega)$ such that $U = u$ in $\Omega \setminus K$.*

*Proof.* Let $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ in a neighborhood of $K$. Then let $u_0 = (1-\varphi)u \in C^\infty(\Omega)$. We shall construct a holomorphic extension $U$ of $u$ such that $U = u_0 - v$, where we need $v \in C^\infty(\Omega)$ and $\overline{\partial} U = 0$. We need

$$\begin{aligned} 0 &= \overline{\partial} U \\ &= \overline{\partial} u - \overline{\partial} v \\ &= \overline{\partial}((1-\varphi)u) - \overline{\partial} v \\ &= (\overline{\partial}(1-\varphi))u - \partial \overline{v} \\ &= -(\overline{\partial}\varphi)u + \overline{\partial} v \end{aligned}$$

with compact support $\subseteq \Omega$, away from $K$. Here, we have used that $u \in \mathrm{Hol}(\Omega \setminus K)$. We have that $(\overline{\partial}\varphi)u \in C_0^\infty(\mathbb{C}^n; \mathbb{C}^n)$. Solve:

$$\overline{\partial} v = -(\overline{\partial}\varphi)u.$$

The compatibility conditions are satisfied:

$$\partial_{\overline{z}_k}\left(\frac{\partial \varphi}{\partial \overline{z}_j} u\right) = \partial_{\overline{z}_j}\left(\frac{\partial \varphi}{\partial \overline{z}_k} u\right) \qquad \forall j, k.$$

So there exists a $v \in C_0^\infty(\mathbb{C}^n)$ solving this, and $\mathrm{supp}(\overline{\partial} v) \subseteq \mathrm{supp}(\varphi)$. So $v = 0$ on the unbounded component $O$ of $\mathbb{C}^n \setminus \mathrm{supp}(\varphi)$. We get $U - u_0 - v = (1-\varphi)u - v \in \mathrm{Hol}(\Omega)$, and $U = u$ on $O \cap (\Omega \setminus \mathrm{supp}\,\varphi))$, which is an open subset of $\Omega \setminus K$. This is nonempty because $\partial O \subseteq \mathrm{supp}(\varphi)$, so since $\Omega \setminus K$ is connected, $U = u$ in $\Omega \setminus K$. □

The following special case is of note:

**Corollary 1.1.** *Let $f \in \mathrm{Hol}(\mathbb{C}^n)$ with $n > 1$. Then $f$ cannot have an isolated zero.*

*Proof.* If $f(0) = 0$ and $f \neq 0$ on $0 < |z| < R$, then apply the Hartogs extension theorem to $K = \{0\}$ and $\Omega = \{|z| < R\}$.Then $h = 1/f \in \mathrm{Hol}(\Omega \setminus K)$, os there exists a extension $U \in \mathrm{Hol}(|z| < R)$. Then $fU = 1$, which is a contradiction. □

## 1.3 Regularization of subharmonic functions

Let $\Omega \subseteq \mathbb{C}$ be open and connected. Let $u \in \mathrm{SH}(\Omega)$ with $u \not\equiv -\infty$. Then $u \in L^1_{\mathrm{loc}}(\Omega)$. Let $0 \le \varphi \in C_0^\infty(\mathbb{C})$ be such that $\mathrm{supp}(\varphi) \subseteq \{|z| < 1\}$ and $\int \varphi(z)\, L(dz) = 1$, where $\varphi$ depends only on $|z|$.

**Remark 1.1.** We can take

$$\varphi(z) = Ch(1-|z|^2), \qquad h(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \le 0. \end{cases}$$

You can check that $h^{(j)}(0) = 0$ for all $j$, so $h \in C^\infty(\mathbb{R})$.

Define

$$u_\varepsilon = u * \varphi_\varepsilon, \qquad \varphi_\varepsilon(z) = \frac{1}{\varepsilon^2}\varphi\left(\frac{z}{\varepsilon}\right),$$

so

$$u_\varepsilon(z) = \int u(z-\zeta)\varphi_\varepsilon(\zeta)\, L(d\zeta), \qquad z \in \Omega_\varepsilon = \{z \in \Omega : \operatorname{dist}(z, \Omega^c) > \varepsilon\}.$$

**Proposition 1.1.** *$u_\varepsilon \in (C^\infty \cap \mathrm{SH})(\Omega_\varepsilon)$, and $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$.*

*Proof.* We have

$$u_\varepsilon(z) = \frac{1}{\varepsilon^2}\int u(\zeta)\varphi\left(\frac{z-\zeta}{\varepsilon}\right) L(d\zeta) \in C^\infty(\Omega_\varepsilon).$$

Check the sub-mean value inequality: First write

$$u_\varepsilon(z) = \int u(z - \varepsilon\zeta)\varphi(\zeta)\, L(d\zeta).$$

If $z \in \Omega_\varepsilon$ and $r$ is small, then since $u$ is subharmonic,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon(z + re^{it})\, dt &= \frac{1}{2\pi}\int_0^{2\pi}\int u(z + re^{it} - \varepsilon\zeta)\varphi(\zeta)\, L(d\zeta)\, dt \\
&\geq \int u(z - \varepsilon\zeta)\varphi(\zeta)\, L(d\zeta) \\
&= u_\varepsilon(z).
\end{aligned}$$

To show that $u_\varepsilon(z) \geq u(z)$, we have

$$\begin{aligned}
u_\varepsilon(z) &= \int u(z + \varepsilon\zeta)\varphi(\zeta)\, L(d\zeta) \\
&= \int_0^\infty \underbrace{\left(\int_0^{2\pi} u(z + \varepsilon r e^{it})\, dt\right)}_{\geq 2\pi u(z)} \varphi(r) r\, dr \\
&\geq \underbrace{\left(2\pi \int_0^\infty \varphi(r) r\, dr\right)}_{=1} u(z).
\end{aligned}$$

□

We will finish the proof next time.